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B. Sc. (Honrs) Part 1Paper 2

Subject Mathematics

Title/Heading of topic :Integration of rational function

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INTEGRATION OF RATIONAL FUNCTIONS USING PARTIAL FRACTIONS

ABSTRACT. We study a technique, called partial fraction decomposition, to find integration of rational functions.

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1. RATIONAL FUNCTIONS

Definition 1.1. *A rational function is the quotient of two polynomial functions.*

For example,

$$\begin{aligned} f(x) &= \frac{2}{(x+1)^3}, \\ g(x) &= \frac{2x+2}{x^2-4x+8}, \\ h(x) &= \frac{x^5+2x^3-x+1}{x^3+5x}. \end{aligned}$$

are rational functions.

The rational function is said to be **proper** if the degree of its numerator is less than that of its denominator. Certainly, other rational functions are **improper**. In the examples above, f and g are proper while h is improper.

An improper rational function can be written as the sum of a polynomial and a proper rational function by using long division.

Example.

$$h(x) = \frac{x^5+2x^3-x+1}{x^3+5x} = x^2 - 3 + \frac{14x+1}{x^3+5x}.$$

It follows that the problem of integrating rational functions becomes the problem of integrating polynomial functions and proper rational functions. Since finding integration of polynomial is not hard, it is sufficient to study integration of proper rational functions.

2. Partial fraction decomposition (linear factors)

Look at the calculation

$$\frac{1}{x-1}+\frac{1}{x+1}=\frac{x+1+x-1}{(x-1)(x+1)}=\frac{2x}{x^2-1}.$$

The inverse process is called partial fraction decomposition; that [illegible] we write

$$\frac{2x}{x^2-1}$$

as a sum of two simpler fractions.

Example. (Distinct linear factors) Decompose

$$\frac{3x-1}{x^2-x-6}$$

Solution.

We first note that the denominator can be rewritten as $(x+2)(x-3)$ (since its roots are -2 and 3). Secondly, we do the partial fraction decomposition as follows:

Assume that

$$\frac{3x-1}{x^2-x-6}=\frac{A}{x+2}+\frac{B}{x-3}=\frac{A(x-3)+B(x+2)}{(x+2)(x-3)}.$$

Thus

$$3x-1=A(x-3)+B(x+2)$$

for all x. Let $x=3$ to get

$$8=5B, B=\frac{8}{5}$$

and let $x=-2$ to get

$$-7=-5A, A=\frac{7}{5}.$$

So,

$$\frac{3x-1}{x^2-x-6}=\frac{7}{5(x+2)}+\frac{8}{5(x-3)}.$$

Example. Find

$$\int\frac{3x-1}{x^2-x-6}dx$$

Solution.

$$\begin{aligned}\int \frac{3x-1}{x^2-x-6}dx &= \int \left[\frac{7}{5(x+2)} + \frac{8}{5(x-3)}\right] dx \\ &= \int \frac{7}{5(x+2)}dx + \int \frac{8}{5(x-3)}dx \\ &= \frac{7}{5}\ln|x+2| + \frac{8}{5}\ln|x-3| + C.\end{aligned}$$

Example. Find

$$\int \frac{5x+3}{x^3-2x^2-3x}dx$$

Solution. We first need to factor the denominator into a product of some linear factors (or polynomials of degree 1)

$$\begin{aligned}x^3-2x^2-3x &= x(x^2-2x-3) \\ &= x(x^2+x-3x-3) \\ &= x[(x^2+x)-(3x+3)] \\ &= x[x(x+1)-3(x+1)] \\ &= x(x+1)(x-3).\end{aligned}$$

Thus, we can write

$$\frac{5x+3}{x^3-2x^2-3x} = \frac{5x+3}{x(x+1)(x-3)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-3}$$

for some constants A, B and C which we need to determine. This equation gives

$$5x+3 = A(x+1)(x-3) + Bx(x-3) + Cx(x+1) \quad \forall x. \tag{2.1}$$

Letting x in (2.1) be equal to 0, we find

$$\begin{aligned}A(0+1)(0-3) &= 5\cdot 0+3 \\ A &= -1.\end{aligned}$$

Similarly, letting $x=-1$ and $x=3$ gives

$$B = \frac{-1}{2}, C = \frac{3}{2}.$$

Thus,

$$\frac{5x+3}{x^3-2x^2-3x} = -\frac{1}{x} - \frac{1}{2(x+1)} + \frac{3}{2(x-3)}$$

and therefore,

$$\int \frac{5x+3}{x^3-2x^2-3x}dx = -\ln|x| - \frac{1}{2}|x+1| + \frac{3}{2}|x-3| + C.$$

□

Example. (Repeated linear factors) Find

$$\int \frac{x}{(x-3)^2}dx$$

Solution. The decomposition takes the form

$$\frac{x}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}.$$

It follows that

$$x = A(x-3) + B.$$

Letting $x = 3$ gives

$$3 = A \cdot 0 + B, \qquad B = 3.$$

Also, we can compare the coefficient of x to get $A = 1$. Thus,

$$\begin{aligned} \int \frac{x}{(x-3)^2}dx &= \int \left[\frac{1}{x-3} + \frac{3}{(x-3)^2}\right] dx \\ &= \ln|x-3| - 3(x-3)^{-1} + C. \end{aligned}$$

□

Example. (some distinct, some repeated linear factors) Find

$$\int \frac{3x^2 - 8x + 13}{(x+3)(x-1)^2}dx$$

Solution. Decompose the fraction in the following way

$$\frac{3x^2 - 8x + 13}{(x+3)(x-1)^2} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

It can be rewritten as

$$\begin{aligned} \frac{3x^2 - 8x + 13}{(x+3)(x-1)^2} &= \frac{A(x-1)^2 + B(x-1)(x+3) + x(x+3)}{(x+3)(x-1)^2} \\ 3x^2 - 8x + 13 &= A(x-1)^2 + B(x-1)(x+3) + C(x+ \end{aligned}$$

Substitution $x = 1, x = -3$ and $x = 0$ yields $C = 2$, $A = 4$ and $B = -1$.

Thus,

$$\begin{aligned} \frac{3x^2 - 8x + 13}{(x+3)(x-1)^2} &= \frac{4}{x+3} - \frac{1}{x-1} + \frac{2}{(x-1)^2}, \\ \int \frac{3x^2 - 8x + 13}{(x+3)(x-1)^2}dx &= 4\ln|x+3| - \ln|x-1| - 2(x-1)^{-1} + C. \end{aligned}$$

□

General rule for decomposing fractions with repeated linear factors in the denominator: for each factor $(ax+b)^k$ of the denominator, there are k terms in the partial fraction decomposition:

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_k}{(ax+b)^k}.$$

3. Partial fraction decomposition (Quadratic factors)

In factoring denominator of a fraction, we may get some quadratic factors, for example x^2+1, that cannot be factored into linear factors.

Example. (a single quadratic factor) Decompose

$$\frac{6x^2-3x+1}{(4x+1)(x^2+1)}$$

and then find its integral.

Solution. We decompose the fraction as

$$\frac{6x^2-3x+1}{(4x+1)(x^2+1)} = \frac{A}{4x+1} + \frac{Bx+C}{x^2+1}$$

Multiplying both sides by $(4x+1)(x^2+1)$ yields

$$6x^2-3x+1 = A(x^2+1) + (Bx+C)(x^2+1).$$

When $x = -\frac{1}{4}$,

$$\begin{aligned} 6\cdot\left(-\frac{1}{4}\right)^2 - 3\left(-\frac{1}{4}\right) + 1 &= A\left[\left(-\frac{1}{4}\right)^2 + 1\right] \\ \frac{6}{16} + \frac{3}{4} + 1 &= \frac{17A}{16} \\ A &= 2. \end{aligned}$$

When $x = 0$ and $x = -1$, we get

$$\begin{aligned} 1 = 2 + C &\Rightarrow C = -1 \\ 4 = 4 + (B-1)5 &\Rightarrow B = 1. \end{aligned}$$

Thus

$$\frac{6x^2-3x+1}{(4x+1)(x^2+1)} = \frac{2}{4x+1} + \frac{x-1}{x^2+1}$$

and

$$\begin{aligned}
\int \frac{6x^2-3x+1}{(4x+1)(x^2+1)}dx &= \int \frac{2}{4x+1}dx + \int \frac{x-1}{x^2+1}dx \\
&= \frac{1}{2}\ln|4x+1| + \int \frac{x}{x^2+1}dx - \int 1x^2+1dx \\
&= \frac{1}{2}\ln|4x+1| + \frac{1}{2}\int \frac{du}{u} - \arctan x \\
&= \frac{1}{2}\ln|4x+1| + \frac{1}{2}ln|u| - \arctan x + C \\
&= \frac{1}{2}\ln|4x+1| + \frac{1}{2}ln(x^2+1) - \arctan x + C.
\end{aligned}$$

Here, we have used the substitution $u = x^2+1$ to find $\int \frac{x}{x^2+1}dx$.

4. Summary

To decompose a rational function

$$f(x) = \frac{p(x)}{q(x)}$$

into partial fractions, proceed as follows

Step 1: If f is improper, divide $p(x)$ by $q(x)$ to obtain

$$f(x) = \text{ a polynomial } + \frac{N(x)}{D(x)}.$$

Step 2 Factor $D(x)$ into a product of linear and irreducible quadratic factors.

Step 3 For each factor of the form $(ax+b)^k$, expect the decomposition as

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_k}{(ax+b)^k}.$$

Step 4 For each factor of the form ax^2+bx+c, expect the decomposition as

$$\frac{Bx+C}{ax^2+bx+c}$$

Step 5 Set $\frac{N(x)}{D(x)}$ equal to the sum of all partial fractions in steps 3 and 4. The number of constants should be equal to the degree of the denominator $D(x)$.

Step 6 Multiply both sides by $D(x)$ and assign convenient values to the variable x to find all constants.